第　一　讲

线性偏微分方程的通解

北京大学　物理学院

数学物理方法课程组

2007年春

讲授要点

1. 线性偏微分方程解的基本性质
   - 解的可叠加性
   - 线性偏微分方程的通解
2. 无界弦上波的传播
   - 一维齐次波动方程的通解
   - d'Lambert解法：行波解

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线性算符L

方程类型	方　　程	线性算符L
波动方程	$\frac{\partial^2 u}{\partial t^2} - a^2\nabla^2 u = f$	$L \equiv \frac{\partial^2}{\partial t^2} - a^2\nabla^2$
热传导方程	$\frac{\partial u}{\partial t} - \kappa\nabla^2 u = f$	$L \equiv \frac{\partial}{\partial t} - \kappa\nabla^2$
Poisson方程	$\nabla^2 u = f$	$L \equiv \nabla^2$

线性性质： $L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2]$

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线性偏微分方程

- 把线性偏微分方程统一写成算符形式

$$L[u] = f$$

- 其中 u 未知函数
  L 线性算符
  f 已知函数，称为方程的非齐次项

- 具有非齐次项的偏微分方程称为非齐次偏微分方程

- 如果$f = 0$，方程就是齐次的

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什么叫“偏微分方程的解”？

如果函数u使方程$L[u]=f$恒成立，则称u是方程$L[u]=f$的解

解的叠加性之一

若u_1和u_2都是齐次方程$L[u]=0$的解

$$L[u_1]=0 \qquad L[u_2]=0$$

则它们的线性组合$c_1u_1+c_2u_2$也是齐次方程的解

$$L[c_1u_1+c_2u_2]=0$$

其中c_1和c_2是任意常数

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则它们的差u_1-u_2一定是相应的齐次方程的解

$$L[u_1-u_2]=0$$

非齐次方程的一个特解加上相应齐次方程的解仍是非齐次方程的解

非齐次方程的通解 = 非齐次方程的任一特解
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解的叠加性之三

若u_1和u_2分别满足非齐次方程

$$L[u_1] = f_1 \qquad L[u_2] = f_2$$

则它们的线性组合$c_1u_1 + c_2u_2$满足非齐次方程

$$L[c_1u_1 + c_2u_2] = c_1f_1 + c_2f_2$$

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- 二阶常微分方程的通解中含有两个任意常数
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- 例如，偏微分方程 $\dfrac{\partial^2 u(x,y)}{\partial x^2} = 0$ 的通解就是
$$u(x,y) = c_1(y) + xc_2(y)$$
其中 $c_1(y)$ 和 $c_2(y)$ 是任意函数
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二维Laplace方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 的通解

作变换 $\xi = x + \mathrm{i}y,\ \eta = x - \mathrm{i}y$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}$$

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☞ 原方程变为 $\frac{\partial^2 u}{\partial \xi\partial \eta}=0$

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$$u(x,y)=f(x+\mathrm{i}y)+g(x-\mathrm{i}y)$$

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能否由偏微分方程的通解出发

来求整个偏微分方程定解问题的解？

- 解常微分方程定解问题时，通常总是先求出微分方程的特解，由线性无关的特解叠加出通解， 而后用定解条件(例如初条件)定出叠加系数

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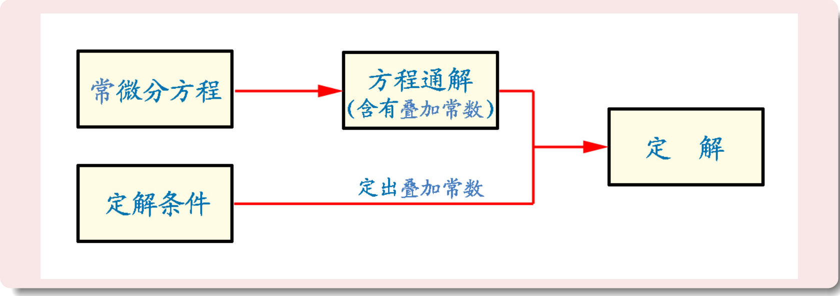

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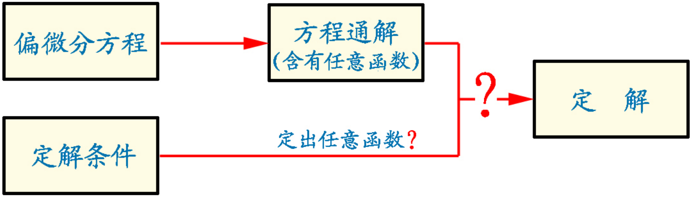

二维Laplace方程
+
边界条件

原则上可以运用
Poisson公式

波动方程？

二维Laplace方程 + 边界条件

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作变换 $\xi = x + at,\ \eta = x - at$

☞ $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

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☞ 原方程变为 $\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$

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$$u(x,t) = f(x-at) + g(x+at)$$

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- 此解式表明，波动方程的通解由两个波组成
- $f(x-at)$代表沿x轴向右传播的波
  当$t=0$时，波形为$f(x)$
  而后以恒定速率a向右传播，保持波形不变
- $g(x+at)$则代表沿x轴向左传播的波
  当$t=0$时，波形为$g(x)$
  而后也以同样的恒定速率a向左传播，保持波形不变

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- ☞ 单独的$f(x-at)$和$g(x+at)$都是波动方程的解．它们独立传播，互不干扰
- ☞ 这正是因为波动方程是线性齐次方程，具有解的叠加性

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讲授要点

1. 线性偏微分方程解的基本性质
   - 解的可叠加性
   - 线性偏微分方程的通解
2. 无界弦上波的传播
   - 一维齐次波动方程的通解
   - d'Lambert解法：行波解

定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad -\infty < x < \infty,\ t > 0$$
$$u(x,t)\Big|_{t=0} = \phi(x), \qquad -\infty < x < \infty$$
$$\frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), \qquad -\infty < x < \infty$$

将通解$u(x,t) = f(x-at) + g(x+at)$代入初始条件

$$f(x) + g(x) = \phi(x)$$
$$a\left[f'(x) - g'(x)\right] = -\psi(x)$$

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将第二式积分

$$f(x) + g(x) = \phi(x)$$

$$f(x) - g(x) = -\frac{1}{a}\int_0^x \psi(\xi)\mathrm{d}\xi + C$$

定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad -\infty < x < \infty, \ t > 0$$

$$u(x,t)\Big|_{t=0} = \phi(x), \qquad -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), \qquad -\infty < x < \infty$$

由此解出

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_0^x \psi(\xi)\mathrm{d}\xi + \frac{C}{2}$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_0^x \psi(\xi)\mathrm{d}\xi - \frac{C}{2}$$

定解问题

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$$u(x,t) = f(x-at) + g(x+at)$$

$$= \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right]$$

$$+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi$$

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- 第一项表示由初位移$u(x,t)\big|_{t=0} = \phi(x)$激发的行波
  以后分成相等的两部分，独立地向左右传播，速率为a
- 第二项表示由初速度$\frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x)$激发的行波
  它将左右对称地扩展到$[x-at, x+at]$ 的范围，传播的速率也是a

$$u(x,t)=\frac{1}{2}\left[\phi(x-at)+\phi(x+at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi$$

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