量子力学期末试题 A 答案和评分

一. (10 分)

5 分 a. $s_x s_z s_x s_y s_x = -s_z s_x^2 s_y s_x = -\frac{\eta^2}{4} s_z s_y s_x = i(\frac{\eta}{2})^5$

或 $= -\frac{\eta^2}{4}\frac{1}{2}(s_y s_z - s_z s_y)s_x = i(\frac{\eta}{2})^5$

5 分

b. $\underline{s} \times \underline{s} = \vec{i}(s_y s_z - s_z s_y) + \vec{j}(s_z s_x - s_x s_z) + \vec{k}(s_x s_y - s_y s_x) = i\eta \underline{s}$

二. (12 分) $\langle\alpha|\alpha\rangle = 1 \quad \therefore |\alpha'\rangle = |\alpha\rangle$

4 分 $|\beta'\rangle = N(|\beta\rangle - |\alpha\rangle\langle\alpha|\beta\rangle) = N(|\beta\rangle - 0.3|\alpha\rangle)$

由 $\langle\beta'|\beta'\rangle = 1 = N^2(\langle\beta| - 0.3\langle\alpha|)(|\beta\rangle - 0.3|\alpha\rangle) = N^2(1 - 0.3^2 \cdot 2 + 0.3^2)$

2 分 $N = \frac{1}{\sqrt{0.91}}$， $|\beta'\rangle = \frac{1}{\sqrt{0.91}}(|\beta\rangle - 0.3|\alpha\rangle)$

4 分 $|\gamma'\rangle = N(|\gamma\rangle - 0.2|\alpha\rangle - |\beta'\rangle\langle\beta'|\gamma\rangle)$

$$\langle\gamma'|\gamma'\rangle = 1 = N^2(\langle\gamma|\gamma\rangle - 0.2\langle\gamma|\alpha\rangle - \langle\beta'|\gamma\rangle\langle\gamma|\beta'\rangle - 0.2\langle\alpha|\gamma\rangle + 0.2\cdot 0.2 - \langle\beta'|\gamma\rangle\langle\gamma|\beta'\rangle + \langle\beta'|\gamma\rangle\langle\gamma|\beta'\rangle)$$

$$\langle\beta'|\gamma\rangle = \frac{1}{\sqrt{0.91}}(\langle\beta|\gamma\rangle - 0.3\langle\alpha|\gamma\rangle) = \frac{0.74}{\sqrt{0.91}}$$

$$N^2 \cdot (1 - 0.2^2 - \frac{0.74^2}{0.91} - 0.2^2 + 0.2^2) = N^2\frac{0.326}{0.91} = 1,$$

2 分 $N = 1.67$

三. (16 分) $\hat{s}_z|m\rangle = m\frac{\eta}{2}|m\rangle$

$$\hat{s}_n|m\rangle' = e^{-i\hat{s}_z\varphi/\eta}e^{-i\hat{s}_y\theta/\eta}\hat{s}_z e^{i\hat{s}_y\theta/\eta}e^{i\hat{s}_z\varphi/\eta}|m\rangle'$$

如 $|m\rangle' = e^{-i\hat{s}_z\varphi/\eta}e^{-i\hat{s}_y\theta/\eta}|m\rangle'$， 则 $\hat{s}_n|m\rangle' = \frac{\eta}{2}m|m\rangle'$

6 分 $\therefore$ 它的本征值为 $\pm\frac{\eta}{2}$

相应的本征值在 $\hat{s}_z$ 表象中的表示

$$\langle m'|m\rangle' = \langle m'|(\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2})(\cos\frac{\theta}{2} - i\sigma_y\sin\frac{\theta}{2})|m\rangle$$

$$\langle m'|(\cos\frac{\varphi}{2}\cos\frac{\theta}{2} - i\sigma_y\cos\frac{\varphi}{2}\sin\frac{\theta}{2} - im\sin\frac{\varphi}{2}\cos\frac{\theta}{2} + i\sigma_x\sin\frac{\varphi}{2}\sin\frac{\theta}{2}|m\rangle$$

$$= \langle m'|\cos\frac{\theta}{2}(\cos\frac{\varphi}{2} - im\sin\frac{\varphi}{2}) + \sin\frac{\theta}{2}(\sigma_- e^{i\varphi/2} - \sigma_+ e^{-i\varphi/2})|m\rangle$$

6 分 $= \cos\frac{\theta}{2} e^{\mu i\varphi/2}\Big|_{m'=m=\pm 1} + \sin\frac{\theta}{2}(\pm)e^{\pm i\varphi/2}\delta\Big|_{\substack{m=1,m'=-1\\ m=-1,m'=1}}$

2 分 $\hat{s}_n$ 本征值为 $\eta/2$，本征表示为 $\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\varphi/2} \\ \sin\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix}$

2 分 $-\eta/2$，本征表示为 $\begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\varphi/2} \\ \cos\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix}$

四. (18 分)

6 分 a. $\varphi_{p_x} = \frac{1}{\sqrt{2\pi\eta}} \int_{-2\pi/b}^{2\pi/b} e^{-ip_x x/\eta} \sqrt{\frac{b}{2\pi}} \frac{e^{ibx} - e^{-ibx}}{2i} dx$

$$= (\frac{b}{\eta})^{1/2} \frac{1}{4i\pi} \int [e^{i(bx - p_x x)/\eta} - e^{-i(bx + p_x x/\eta)}]dx$$

$$= \frac{1}{4\pi i} \frac{b}{\eta} [\frac{1}{i(b - p_x/\eta)} e^{i(b - p_x/\eta)x}\Big|_{-2\pi/b}^{2\pi/b} + \frac{1}{i(b + p_x/\eta)} e^{-i(b + p_x/\eta)x}\Big|_{-2\pi/b}^{2\pi/b}]$$

$$= \frac{1}{4\pi}(\eta b)^{1/2}(+2i)\sin\frac{2\pi p_x}{\eta b} \frac{2\eta b}{(\eta b)^2 - p_x^2}$$

该态中粒子动量可能测得值为 $-\infty < p_x < \infty$

5 分 b. $\frac{d|\varphi(p_x)|^2}{dp_x} = 0 = \frac{d}{dp_x}\{\sin^2\frac{2\pi p_x}{\eta b} \frac{1}{[(\eta b)^2 - p_x^2]^2}\}$

$$\therefore \quad \frac{4\pi}{\eta b}\cos\frac{2\pi p_x}{\eta b} + \sin\frac{2\pi p_x}{\eta b} \frac{4p_x}{(\eta b)^2 - p_x^2} = 0$$

$$[(\eta b)^2 - p_x^2]\cos\frac{2\pi p_x}{\eta b} + \frac{p_x \eta b}{\pi}\sin\frac{2\pi p_x}{\eta b} = 0$$

$\therefore$ 有解 $p_x = \pm\eta b$

3 分 c. $\varphi(p_x)\big|_{\eta b} = \frac{i}{\pi}(\eta b)^{3/2} \frac{\frac{2\pi}{\eta b}\cos\frac{2\pi p_x}{\eta b}}{-2p_x}\Bigg|_{\eta b}$

发现粒子在 $\eta b - \eta b + dp_x$ 区间中的几率为

$$|\varphi(\eta b)|^2 dp_x = \frac{1}{\eta b} dp_x$$

4 分　d. $\psi(x,t)=\int\varphi(p_x)\frac{1}{(2\pi\eta)^{1/2}}e^{ip_x/\eta-i\frac{p_x^2}{2m}t/\eta}dp_x$

五. (18 分)

a.　2 分　$\varepsilon_n=(n+\frac{1}{2})\eta\omega$,

3 分　$E_{基}=\frac{5}{2}\eta\omega$,　$E_1=\frac{7}{2}\eta\omega$

基态 $n_0=2$，$n_1=1$

2 分　$$\psi_{01}=\frac{1}{\sqrt{3!}}\begin{vmatrix}u_0(1)\alpha(1) & u_0(2)\alpha(2) & u_0(3)\alpha(3)\\ u_0(1)\beta(1) & u_0(2)\beta(2) & u_0(3)\beta(3)\\ u_1(1)\beta(1) & u_1(2)\beta(2) & u_1(3)\beta(3)\end{vmatrix}$$

$$=\frac{1}{\sqrt{3}}[u_0(1)u_0(2)\chi_{00}(12)u_1(3)\alpha(3)-u_0(1)u_0(3)\chi_{00}(13)u_1(2)\alpha(2)$$
$$+u_0(2)u_0(3)\chi_{00}(23)u_1(1)\alpha(1)]$$

1 分　$$\psi_{02}=\frac{1}{\sqrt{3}}[u_0(1)u_0(2)\chi_{00}(12)u_1(3)\beta(3)$$
$$-u_0(1)u_0(3)\chi_{00}(13)u_1(2)\beta(2)$$
$$+u_0(2)u_0(3)\chi_{00}(23)u_1(1)\beta(1)]$$

第一激发态 $n_0=2$，$n_2=1$

2 分　$$\psi_{11}=\frac{1}{\sqrt{3}}[u_0(1)u_0(2)\chi_{00}(12)u_2(3)\alpha(3)$$
$$-u_0(1)u_0(3)\chi_{00}(13)u_2(2)\alpha(2)$$
$$+u_0(2)u_0(3)\chi_{00}(23)u_2(1)\alpha(1)]$$

1 分　$$\psi_{12}=\frac{1}{\sqrt{3}}[u_0(1)u_0(2)\chi_{00}(12)u_2(3)\beta(3)$$
$$-u_0(1)u_0(3)\chi_{00}(13)u_2(2)\beta(2)$$
$$+u_0(2)u_0(3)\chi_{00}(23)u_1(1)\beta(1)]$$

2 分　$$\psi_{13}=\frac{1}{\sqrt{3}}[u_1(1)u_1(2)\chi_{00}(12)u_0(3)\alpha(3)$$
$$-u_1(1)u_1(3)\chi_{00}(13)u_0(2)\alpha(2)$$
$$+u_1(2)u_1(3)\chi_{00}(23)u_0(1)\alpha(1)]$$

1 分　$$\psi_{14}=\frac{1}{\sqrt{3}}[u_1(1)u_1(2)\chi_{00}(12)u_0(3)\beta(3)$$
$$-u_1(1)u_1(3)\chi_{00}(13)u_0(2)\beta(2)$$
$$+u_1(2)u_1(3)\chi_{00}(23)u_0(1)\beta(1)]$$

b.　4 分　基态二重简并

第一激发态四重简并

六. (16 分)

3 分　粒子的能量为 $\frac{\pi^2\eta^2}{2ma^2}(n_x^2+n_y^2+n_z^2)$

第一激发态为 1 1 2

1 2 1

2 1 1

$$E_1^0=\frac{\pi^2\eta^2}{2ma^2}(1+1+4)=\frac{3\pi^2\eta^2}{\pi a^2},$$

5 分　$\langle\vec{r}|1\rangle=(\frac{2}{a})^{3/2}\sin\frac{\pi}{a}x\sin\frac{\pi}{a}y\sin\frac{2\pi}{a}z$

$$\langle\vec{r}|2\rangle=(\frac{2}{a})^{3/2}\sin\frac{\pi}{a}x\sin\frac{2\pi}{a}y\sin\frac{\pi}{a}z$$

$$\langle\vec{r}|3\rangle=(\frac{2}{a})^{3/2}\sin\frac{2\pi}{a}x\sin\frac{\pi}{a}y\sin\frac{\pi}{a}z$$

$$\langle 1|\hat{H}'|1\rangle=(\frac{2}{a})^2\int_0^a x\sin^2\frac{\pi}{a}xdx\cdot\int_0^a y\sin^2\frac{\pi}{a}ydy$$

$$\int_0^a x\sin^2\frac{\pi}{a}xdx=\frac{a^2}{4}$$

$$\therefore\langle 1|H'|1\rangle=(\frac{2}{a})^2\cdot\frac{a^2}{4}\cdot\frac{a^2}{4}\cdot b=\frac{1}{4}ba^2$$

$$\langle 1|H'|2\rangle=\langle 1|H'|3\rangle=0$$

$$\langle 2|H'|2\rangle=(\frac{2}{a})^2 b\int_0^a x\sin^2\frac{\pi}{a}xdx\cdot\int_0^a y\sin^2\frac{2\pi}{a}ydy=\frac{1}{4}ba^2$$

$$\langle 2|H'|3\rangle=(\frac{2}{a})^2 b\int_0^a x\sin\frac{\pi}{a}x\sin\frac{2\pi}{a}xdx\cdot\int_0^a y\sin\frac{2\pi}{a}y\sin\frac{\pi}{a}ydy$$

$$=(\frac{2}{a})^2 b(-\frac{8a^2}{9\pi^2})(-\frac{8a^2}{9\pi^2})=\frac{4ba^2\cdot 64}{81\pi^4}$$

$$\langle 3|H'|3\rangle=(\frac{2}{a})^2 b\int_0^a x\sin^2\frac{2\pi}{a}xdx\cdot\int_0^a y\sin^2\frac{\pi}{a}ydy=\frac{1}{4}ba^2$$

4 分　于是有：$\begin{vmatrix}\frac{1}{4}ba^2-E^1 & 0 & 0\\ 0 & \frac{1}{4}ba^2-E^1 & \frac{64\cdot 4ba^2}{81\pi^4}\\ 0 & \frac{64\cdot 4ba^2}{81\pi^4} & \frac{1}{4}ba^2-E^1\end{vmatrix}=0$

2 分　$\therefore$　$E_1^1=\frac{1}{4}ba^2$

2 分　$E_{2,3}^1=\frac{1}{4}ba^2\pm\frac{64\cdot 4ba^2}{81\pi^4}=(\frac{1}{4}\pm\frac{64\cdot 4}{81\pi^4})ba^2=[\frac{1}{4}\pm(\frac{4}{3\pi})^4]ba^2$